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Efficient Formation of Element Stiffness Matrices on Personal Computers, using Finite-Element Methods

Mohamed M. El-Awad

Department of Mechanical Engineering, Faculty of Engineering, University of Khartoum, Khartoum, Sudan (Tel.: +249-911700885; E-mail: mmelawad@uofk.edu)

Abstract: This paper compares one-point quadrature with analytical integration as efficient alternative integration techniques to standard two-point Gauss-Legendre quadrature for finite-element codes that adopt bi-linear quadrilateral elements. The accuracy of the solutions obtained by the two alternative schemes for the heat-conduction problem on a square domain was compared with that obtained by standard two-point quadrature. The results show that one-point quadrature saves 75.0% of computer time compared to the two-point quadrature scheme while analytical integration saves 37.5% of computer time. Although, one-point quadrature generally requires an “hour-glass” correction, the paper shows that such a correction is not necessary when a Dirichlet boundary condition is applied over a large part of the solution domain. Solution of the conduction problem on a skewed domain proves that one-point quadrature remains acceptably accurate even for highly distorted elements.

Keywords: Finite-element method; Gauss-Legendre quadrature; One-point quadrature; Hour-glass correction; Analytical integration; Personal computers.

1. INTRODUCTION

Scientific and engineering problems often involve the solution of ordinary or partial differential equations that cannot be obtained analytically. Therefore, the equations are solved using computer-based numerical methods, such as the finite element (FE) method and the finite-volume (FV) method. These numerical methods transform the given differential equation into an algebraic system, the solution of which yields approximate values of the dependent variable (e.g. temperature) at selected values of the independent variable (e.g. space or time). The availability of powerful and cheap personal computers (PCs) nowadays, allowed scientists and engineers in almost all fields to take advantage of the computer-based numerical software. However, the type and size of the problems that can be solved on PCs are still limited compared to modern mainframe computers and supercomputers which are much faster. Supercomputers also have parallel and vector processing capabilities and by taking advantage of these special features the computational speed can be increased by orders of magnitude while reducing the storage requirement of the numerical solution. Personal computers, which do not have such capabilities, require special techniques in order to minimise their computation requirements.

The need to adopt efficient numerical techniques appears clearly when applying the FE method on a PC. Compared to other numerical methods, the FE method offers greater flexibility in handling difficult geometrical and boundary conditions but its running time is high. The most time-consuming step in the FE method is usually the formation of the elemental matrices that require numerical integration at element-level. For the bi-linear element, which is a commonly used element in FE codes, the standard two-point Gauss-Legendre quadrature, is time consuming. In order to minimise the computational time, Gresho and Sani [1] and Molina and Huot [2] used one-point quadrature. Compared to the two-point quadrature, one-point quadrature reduces the computation time by a factor of 4 for 2D elements and a factor of 8 for 3D elements. However, in fluid-flow and heat-transfer problems one-point-quadrature solutions exhibit an oscillatory behaviour known as "hour-glass" mode. Gresho and Sani [1] added an “hour-glass” correction term to suppress the oscillations which result from under-integration of the diffusion term in the governing equation. As an alternative to one-point quadrature, Mizukami [3] suggested analytic-integration formula which are exact only for the parallelogram bi-linear element but can be used as a good approximation for the general quadrilateral element. Although he showed that the analytical integration does not produce the characteristic oscillatory behaviour of one-point quadrature, he did not report the values of the computation time.

This paper compares the accuracy and computer time of one-point quadrature and analytical integration by solving the heat conduction problem on a personal computer using bi-linear quadrilateral elements. The governing equation for the heat conduction problem, Laplace equation, is met in many other engineering and scientific applications. A brief description of the FE method is given in Section 2 and the alternative integration techniques are described in Section 3. Section 4 considers the solution of the heat conduction problem in a square plate where undistorted elements can be used. The problem associated with one-point quadrature, and the effect of hour-glass correction, is also discussed in this section. Section 5 deals with the issue of element distortion by solving the heat-conduction problem on a skewed domain.

2. THE FINITE-ELEMENT METHOD

The finite element method has its origin in the aerospace industry where it was used to study stresses in complex airframe structures [4,5]. However, the true potential of the method became apparent when it was later extended by using the calculus of variations and the weighted-residual methods to become a general numerical method for solving field equations. Field equations that govern many scientific and engineering problems can be written as:

$$D(u) = q \qquad \text{in } \Omega \tag{1}$$

where, D is the differential operator, u the dependent variable, q a function that specifies the prescribed boundary conditions, and Ω the solution domain. Fig. (1.a) shows the domain of the problem with three types of possible boundary conditions. At the part of the boundary (A-B and E-F) where certain values of u are known, the boundary condition is known as a Dirichlet type. At the part of the boundary (B-C-D-E) where the normal gradient of u is specified, the boundary condition is known as a flux type. At the part of the boundary (A-F) where the flux is zero, the boundary condition is known as a Neumann type.

The finite element method is a numerical analysis technique for obtaining approximate solutions to Eq. (1) when an exact analytical solution cannot be found. Two popular formulations of the method are the Ritz and the Galerkin formulations. The Ritz formulation requires the differential equation Eq. (1) to be converted into an integral form using calculus of variation. Sometimes the integral form is easily derivable from the physics of the problem, but for many practical engineering problems finding the variational form of the equation is difficult. A good example of this is the Navier-Stokes equations that govern fluid-flows. To solve such problems, scientists and engineers use Galerkin formulation, which applies the weighted-residual methods. Accordingly, the solution domain is divided into small regions called "elements", as shown on Fig. (1.b). The elements are connected with "nodes". The variation of the dependent variable u in each element is represented by a simple function $\tilde{u}$ that approximates u within the element. Substituting the approximate primary variable $\tilde{u}$ in Eq. (1), results in a residue (R) that depends on the approximating function, i.e.,

$$D(\tilde{u}) - q = R \tag{2}$$

If the residue R can be made equal zero everywhere, then the approximate solution becomes equal to the true value. Since it is very difficult to make the residue 0 at all points, a weighted residual is made equal to zero, i.e.,

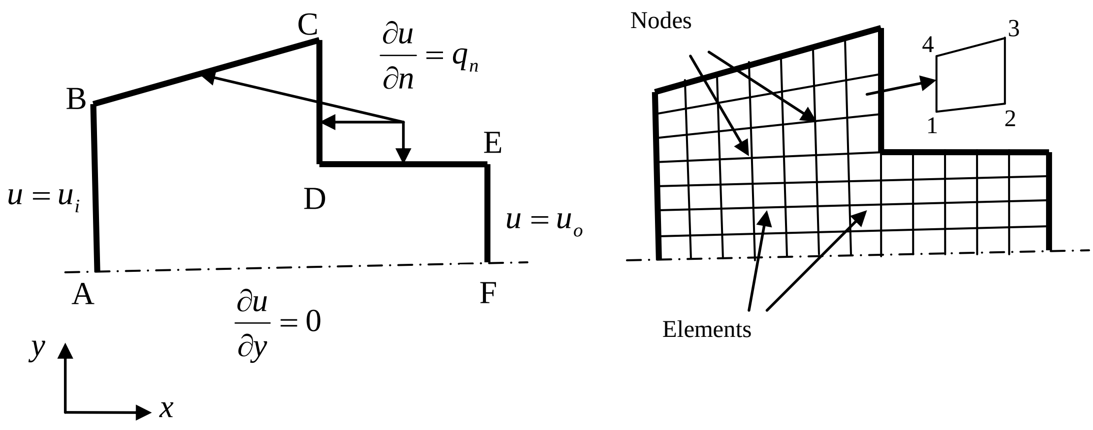
$$\int_{\Omega} wR\,\mathrm{d}A = 0 \tag{3}$$

where w is a weighting function and dA is the area of the element.

In the Galerkin method, the weighting function w has the same mathematical form as the approximating function $\tilde{u}$. The unknown coefficients of the function are replaced by the unknown nodal values of $\tilde{u}$:

$$\tilde{u} = [N]\{u^{ne}\} \tag{4}$$

where $[N]$ is the matrix of shape functions and $\{u^{ne}\}$ is the nodal values of $\tilde{u}$.



(a) (b)

Fig. 1. The problem domain and its finite-element representation

Substituting Eq. (4) in Eq. (1) leads to an algebraic system of equations in terms of the elemental values of $\tilde{u}$. The elemental system of algebraic equations takes the form:

$$[k]^{[e]}\{\tilde{u}\}^{[e]} = \{f\}^{[e]} \qquad (5)$$

where $[k]^{[e]}$ is called the element stiffness matrix, $\{\tilde{u}\}^{[e]}$ is the vector containing the values of the unknown field variable at element nodes, and $\{f\}^{[e]}$ is the corresponding force vector. The sizes of $[k]^{[e]}$, $\{\tilde{u}\}^{[e]}$, and $\{f\}^{[e]}$ are mxm, mx1, and mx1, respectively, where m is the number of nodes in the element. A global algebraic system is then formed by assembling the smaller systems given by Eq. (5). The algebraic system thus obtained takes the form:

$$[k]\{\tilde{u}\} = \{f\} \qquad (6)$$

where $[k]$ is the global stiffness matrix, $\{\tilde{u}\}$ is the global vector that contains the solution of the governing partial differential equation at selected points in space or time, and$\{f\}$ is the global "force" vector. The boundary conditions are then applied to the assembled system and the system is solved by a suitable solver.

3. FORMATION OF THE ELEMENT STIFFNESS MATRIX

Entries of the element stiffness matrix, $k^{[e]}$, are obtained by evaluating some integrals over the element which involve the shape functions and/or their derivatives such as [4]:

$$\int_{[e]} N_1(x)\,N_2(x)\,dx\,,\ \int_{[e]} \frac{dN_1}{dx}\frac{dN_2}{dx}\,dx$$

where N_1(x) and N_2(x) are the two shape functions associated with node 1 and node 2 of a one-dimensional linear element. For two-dimensional problems, the shape functions are functions of both x and y. Accordingly, the integral over the element takes the following form:

$$k_{ij}^{[e]} = \int_x\int_y p(x,y)\partial x\partial y \qquad (7)$$

where, $p(x,y)$ is a polynomial of a degree that depends on the order and shape of the element used as well as the governing equation itself. Except for elements of low orders and simple shapes, explicit analytic expressions are not available for the above integral, which has to be evaluated numerically.

Gauss-Legendre quadrature is the numerical integration method usually adopted in FE codes for evaluating the integrals in Eq. (7), because of its efficiency compared to other techniques. Accordingly, the integral is first transformed into the local (ζ,η) plane, where $-1\le\zeta\le1$ and $-1\le\eta\le1$. The integral then becomes:

$$k_{ij}^{[e]} = \int_{-1}^{+1}\int_{-1}^{+1} p(\zeta,\eta)\ \mathrm{Abs}|J|\ \partial\zeta\partial\eta \qquad (8)$$

where J is the Jacobian matrix of transformation and $|J|$ its determinant. Introducing a new function $g = \mathrm{Abs}|J|\times p(\zeta,\eta)$ and applying Gauss-Legendre quadrature, the integral is then evaluated as:

$$k_{ij}^{[e]} \approx \sum_{j=1}^{2}\ \sum_{i=1}^{2}\ w_i\, w_j\, g(\zeta_i,\eta_j) \qquad (9)$$

where $g(\zeta_i,\eta_j)$ are values of the function $g(\zeta,\eta)$ at the sampling points and w_i, w_j are the corresponding weighting values. Gauss-Legendre quadrature with n sampling points is exact for polynomials of degree up to 2n-1. Since two-point quadrature (2PQ) is exact for polynomials of up to degree 3, it is usually used for the bi-linear quadrilateral or triangular elements used in most finite-element codes. For two-dimensional problems, 2PQ requires four sampling points (two in each direction) and for three-dimensional problems, it requires eight sampling points. Therefore, the computation of the element systems and their assembly into the global system is usually the most time consuming step in the finite element solution.

3.1 One-Point Quadrature

One-point quadrature (1PQ) evaluates the element integrals at a single point, which is at the centroid of the element. The weighting constants in Gauss-Legendre quadrature w_i and w_j both take the value of 2. The integral in Eq. (9) is then evaluated as:

$$k_{ij}^{[e]} \approx 2\mathrm{x}2\ g(\zeta_o,\eta_o), \qquad (10)$$

where $g(\zeta_o,\eta_o)$ is the value of $g(\zeta,\eta)$ at the single sampling point. Since $g(\zeta,\eta)$ is evaluated once, 1PQ reduces the computational time of 2D problems by a factor of four and that of 3D problems by a factor of eight. However, the penalty for this saving in time could be a reduction in accuracy. Moreover, 1PQ solutions are contaminated by spurious oscillations, or "wiggles", caused by the reduced integration of diffusion-like terms [1,6,7]:

$$k_{ij}^{[e]} = \int_x\int_y \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\partial x\partial y \qquad (11)$$

To suppress these oscillations in the 1PQ solution, Gresho and Sani [1] added the following hour-glass correction (HGC) to the element diffusion matrix:

$$H_{ij}^{[e]} = c\,\underline{x}.\underline{x}^{\mathrm{T}}\ k_{ij}^{[e]} \qquad (12)$$

where, $k_{ij}^{[e]}$ is the element diffusion matrix generated via one-point quadrature, $\underline{x}$ is a vector with four elements, $\underline{x}^{\mathrm{T}}$ = [1,-1,1,-1] its transpose, and c is a tuning constant. It should also be mentioned that reduced integration is used for purposes other than its computational saving. For example, in non-linear elastodynamics analysis, full Gauss-Legendre quadrature is known to be subjected to volumetric locking for incompressible or nearly incompressible materials. For this case, Duarte Filho and Awruch [6] and Duarte Filho *et al*. [7] used 1PQ to prevent volumetric locking.

3.2 Analytical Integration

For simple cases, the integral in Eq. (7) can be obtained analytically. Mizukami [3] showed that, for the bilinear quadrilateral element, the elemental integrals could be expressed explicitly in terms of nodal values by using the following three functions:

$$\alpha(x_k,y_k) = (x_3 - x_1)(y_4 - y_2) - (x_4 - x_2)(y_3 - y_1) \quad (13.a)$$
$$\beta(x_k,y_k) = (x_4 - x_3)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_3) \quad (13.b)$$
$$\gamma(x_k,y_k) = (x_1 - x_4)(y_3 - y_2) - (x_3 - x_2)(y_1 - y_4) \quad (13.c)$$

where, the suffices 1 to 4 refer to the elements nodes (refer to Fig. 1.b). Using the above formulae, the element stiffness matrix for the diffusion term could be calculated from:

$$k_{ij}^{[e]} = \int_x\int_y \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\partial x \partial y =$$
$$\left\{\frac{1}{2}\alpha(u_k, y_k)\alpha(x_k, u_k) + \frac{1}{6}\beta(u_k, y_k)\beta(x_k, u_k) + \frac{1}{6}\gamma(u_k, y_k)\gamma(x_k, u_k)\right\} / \alpha(x_k, y_k) \quad (14)$$

Although Mizukami's formulae are only exact for a parallelogram element, they can be used as a good approximation of the general four-node quadrilateral element. Mizukami [3] compared the accuracy of his analytical integration (AI) formula, with that of one-point quadrature by solving the transient heat conduction equation in a circular plate. His results showed that AI did not produce the oscillatory behaviour associated with 1PQ, but the computation times were not given. The limitation of analytical integration for more general elements is that exact analytical formulae cannot be obtained easily. Therefore, AI did not find a wide application in FE codes. However, with the development of computer-based symbolic integration software such as MAPLE there has been renewed interest in AI [8,9]. Videla et al [8] obtained a saving in computer time of more than 50% by using AI instead of Gauss-Legendre quadrature for evaluating the stiffness matrix of a 8-noded plane quadrilateral sub-parametric finite element. They also conducted a sensitivity analysis to show that their AI formulae lead to more accurate results compared to Gauss-Legendre quadrature even in the case of highly distorted elements.

3. HEAT-CONDUCTION IN A SQUARE PLATE

The two-dimensional heat conduction problem, used here to compare the accuracy and computer time of the alternative integration methods, is governed by the following equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad (15)$$

where, x and y refer to the two coordinates, t to time, and T to temperature. At steady state, the problem is reduced to:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (16)$$

Equation (16) is Laplace equation which is met in different engineering applications. The equation was solved on the square plate of unit width shown on Fig. (2.a) which also shows the boundary conditions.

The problem was first solved for the case when the temperature is specified at a single point, the bottom left corner. A uniform 10x10 grid was used as shown on Fig. (2.b). The numerical scheme, which followed that of Gresho and Sani [1], used four-node, isoparametric elements for the spatial discretization and the explicit forward Euler scheme for the time integration. Fig. (3) shows the solutions obtained by the alternative integration methods after 30 iterations. The figure shows the solutions obtained for this case with 2PQ, AI and 1PQ after 30 solution steps. As the figure suggests, the solution with 1PQ exhibited an unrealistic oscillatory behaviour that was not exhibited by the analytical integration formula given by Mizukami [3]. Fig. (3) also shows the results of 1PQ with hour-glass correction. The oscillation-free solution shown on Fig. (3.d) was obtained by taking the value of the tuning constant c in Eq. (12) as 0.1.

Other tests with modified boundary conditions showed that the oscillations in the 1PQ solution diminished gradually as more nodes at the bottom (or left) boundary of the solution domain were assigned Dirichlet values without adding the

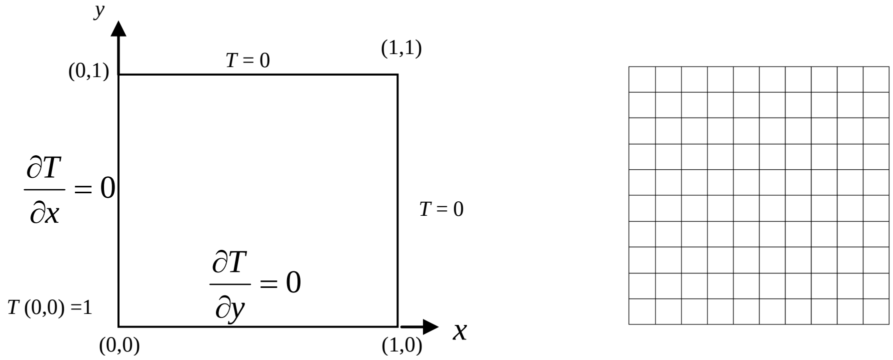
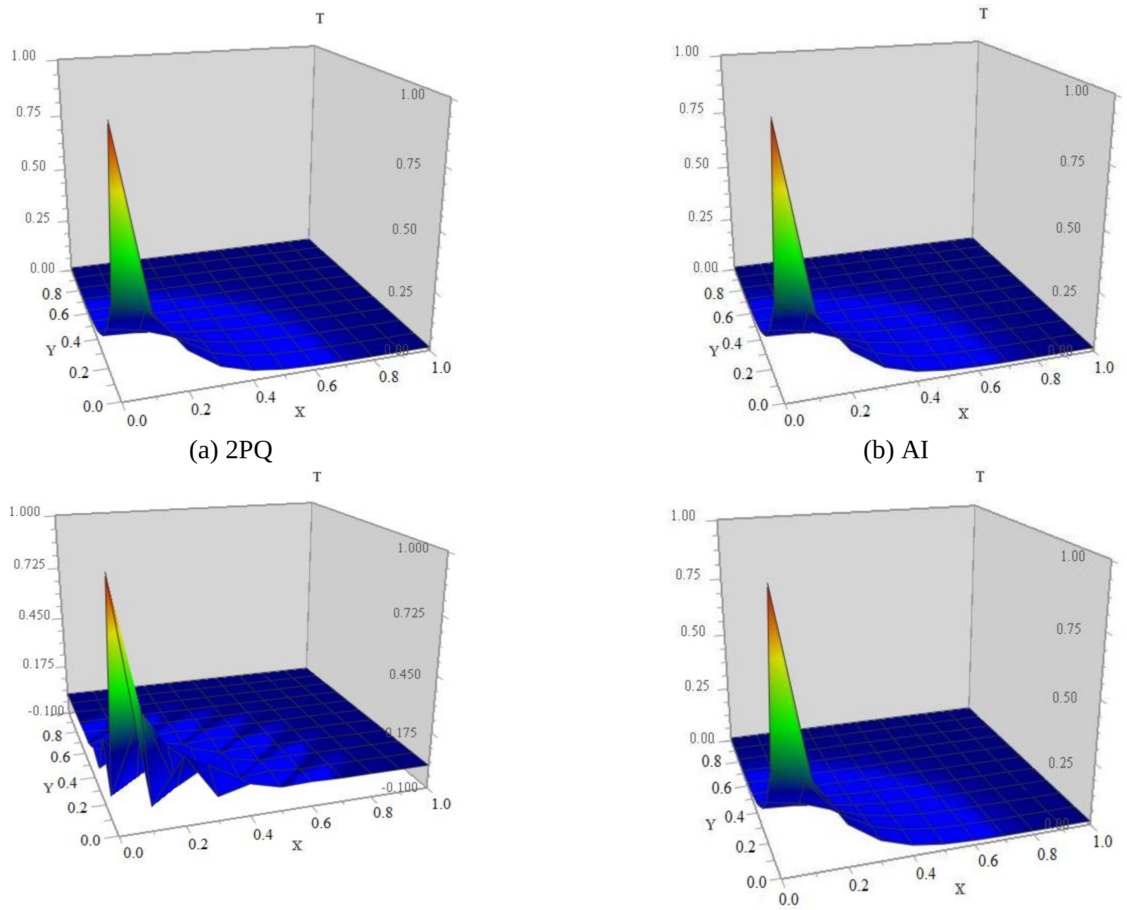


Fig. 2. Heat conduction problem: (a) boundary conditions, (b) FE grid



(a) 2PQ (b) AI

(c) 1PQ (d) 1PQ + HGC

Fig. 3. Heat-conduction problem with the boundary condition $T = 1.0$ at (0,0): solutions with (a) 2PQ, (b) AI, (c) 1PQ without HGC and (d) 1PQ with HGC.

hour-glass correction. Figure (4) shows the results for the case in which temperature varied linearly with x and y at the bottom side and the left side of the plate. The figure shows a good agreement between the solutions obtained with AI and 1PQ, with and without hour-glass correction, with that of the solution obtained with the standard 2PQ.

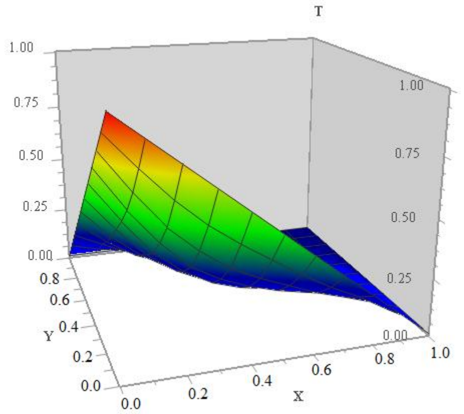
On a PC with a 1333 MHz mobile Intel® Celeron™ processor, the CPU times for 2PQ, AI and 1PQ were 8.0×10^{-2}, 5.0×10^{-2}, and 2.0×10^{-2} seconds, respectively. The added computer time due to the hour-glass correction was insignificant. These figures show that the computer time for the analytical integration is comparable to that of the standard two-point quadrature, but one-point quadrature reduced the time by a factor of four.

5. HEAT-CONDUCTION IN A SKEWED PLATE

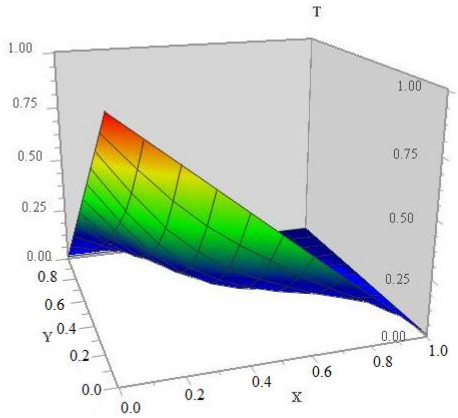
Because of regular geometry, the FE solution applied undistorted elements in solving the test case considered above. To compare the accuracy of one-point quadrature to that of two-point quadrature when distorted elements are used, the steady heat conduction equation (Eq. 16) was solved on the skewed domain shown on Fig. (5). As happened before, the edges of the plate had unit lengths, but the height H, given by $H = \sin(\theta)$ decreased with decreasing θ (Fig. 5.b). A linear variation of T from 1 to 0 was imposed on both the bottom and left boundaries of the domain, while zero values were assigned at both the upper boundary and right boundary of the domain. The problem was solved for different values of the angle θ on a 10x10 grid. Fig. (6) which compares the solutions obtained at $\theta = 30^o$ shows that the solution with 1PQ, with and without HGC, agreed well with that obtained with 2PQ.

6. CONCLUDING REMARKS

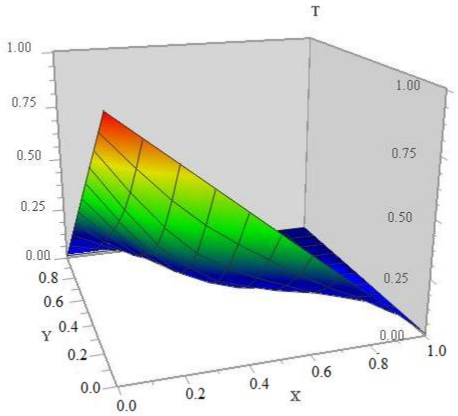
For the quadrilateral bi-linear element which is commonly used in finite-element codes, the present paper shows that one-point quadrature is favourable over analytical integration as an efficient alternative to standard two-point Gauss-Legendre quadrature. While the computer time for the analytical integration is comparable to that of the standard two-point quadrature, one-point quadrature reduced the computer time by a factor of four. Laplace equation, which is considered here, is met in many engineering and scientific applications. Therefore, the present analysis is also relevant to these applications. However, it should be mentioned that the two test cases considered here applied Dirichlet or Neumann type of boundary conditions which did not require evaluation. For problems with specified non-zero heat flux at the boundary, calculating the contribution from boundary conditions with one-point quadrature may reduce the accuracy of the FE method. Since boundary integrals do not usually



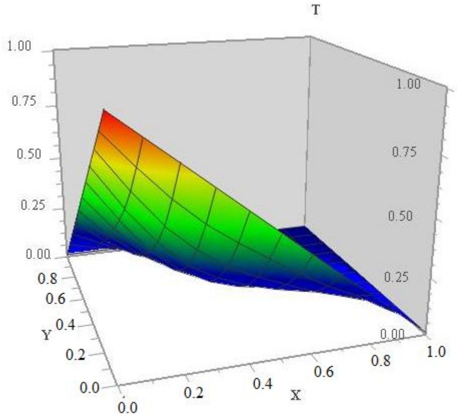
(a) 2PQ



(b) AI

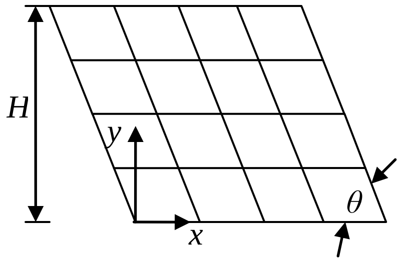


(c) 1PQ

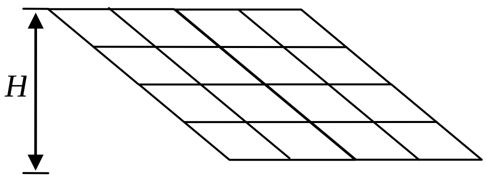


(d) 1PQ+HGC

Fig. 4. Heat-conduction problem with boundary condition (T = 1-x) at y=0 and (T = 1-y) at x=0: solutions with (a) 2PQ, (b) AI, (c) 1PQ without HGC and (d) 1PQ with HGC.

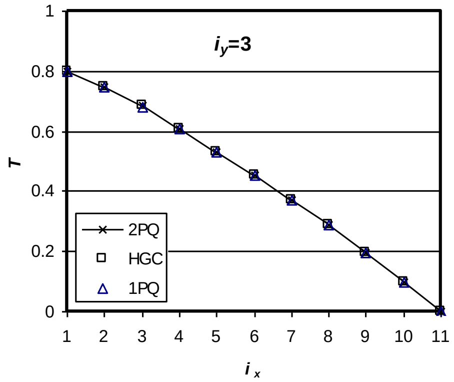


(a) $\theta \approx 60^o$



(b) $\theta \approx 45^o$

Fig. 5. Heat conduction problem in a skewed domain



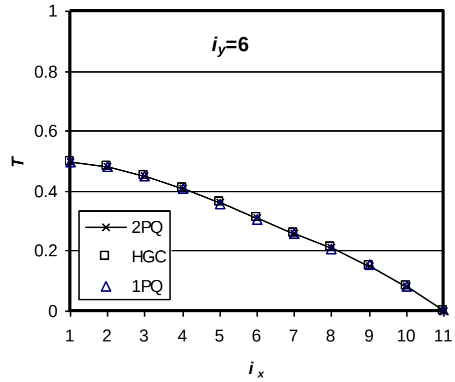


Fig. 6. Solutions of the heat conduction problem in a skewed domain

contribute much to the total computation time, two-point quadrature can be used in their calculation for such problems. In this regard, the study suggests that an important advantage of 1PQ over AI is that it allows the selective use of 2PQ and 1PQ with minor modifications to existing finite-elements codes that use the standard 2PQ.

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