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Estimating the Normalized Polarizability of a Dusty Medium by Using Elliptic Integrals

Sami M. Sharif[1*]
[1]Department of Electrical and Electronics Engineering, University of Khartoum. Email: smsharif@uofk.edu
[*]*Corresponding author (E-mail: smsharif@uofk.edu)*

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ABSTRACT

Microwave signal degradation during dust storms poses significant challenges for high-frequency communication systems, especially in arid regions. Traditional models often assume spherical dust particles, which limits their accuracy in representing real-world propagation effects. This paper presents a theoretical model to estimate the normalized polarizability of dusty media by incorporating non-spherical particle geometries. Specifically, dust particles are modeled as ellipsoids, and their geometry factor is analytically determined using elliptic integrals. The model integrates this shape-dependent factor into the Maxwell-Garnett effective medium framework to evaluate the dielectric properties of dusty environments. Key contributions include: (1) the derivation of depolarization factors using elliptic integrals for arbitrary ellipsoidal geometries, (2) a shape-sensitive dielectric model accounting for orientation and volume fraction, and (3) a parametric study demonstrating the impact of axial ratios and dielectric contrast on normalized polarizability across microwave frequency bands. This work provides a compact analytical formulation that enhances the prediction of electromagnetic behavior in dusty atmospheric conditions, offering practical value for communication system design.

I. INTRODUCTION

Microwave signal propagation through dusty or particulate media has become increasingly important due to the rising use of high-frequency communication systems in arid and semi-arid regions. Dust storms, common in such environments, can severely impair signal quality through scattering, absorption, attenuation, and polarization effects [1–3]. Despite its importance, the modeling of dusty media remains a challenge—primarily due to oversimplified assumptions about particle geometry and material properties [4].

Most existing models assume spherical dust particles and use isotropic dielectric approximations, such as Rayleigh or Mie theory, or apply standard mixing rules like Maxwell-Garnett and Bruggeman formulations [5, 6]. However, these approaches do not accurately capture the electromagnetic behavior of real-world dust particles, which are often non-spherical, randomly oriented, and exhibit complex dielectric responses [7]. Such assumptions lead to inaccurate estimates of the effective dielectric constant and, consequently, poor predictions of signal degradation [8].

To address these limitations, this study develops an analytical framework that incorporates the shape anisotropy of dust particles by modeling them as ellipsoidal inclusions. The core innovation is the use of elliptic integrals to derive depolarization factors and compute a geometry-dependent expression for the normalized polarizability [9], [10]. This allows for the realistic estimation of the dielectric response of dusty media while maintaining analytical tractability.

The significance of the developed expression lies in its ability to capture how particle shape, orientation, and dielectric contrast affect the polarization response of the medium. By extending the Maxwell-Garnett model to account for ellipsoidal particle geometries, this work offers a more accurate and comprehensive tool for predicting electromagnetic wave behavior in dusty atmospheric conditions. The resulting model is particularly valuable for designing robust satellite links, terrestrial microwave systems, and radar applications operating in dusty environments [11], [12].

2. RELATED WORK

The interaction between electromagnetic (EM) waves and particulate media—particularly in dusty or sand-laden environments—has been extensively investigated due to its implications for radar, satellite, and terrestrial wireless communication systems. These environments are especially critical in desert and semi-arid regions, where dust storms can dramatically degrade link performance through absorption, scattering, and depolarization [2, 3]. However, accurately modeling the dielectric response of such media remains a complex challenge [13].

Early models, such as Mie theory and Rayleigh scattering approximations, provided analytical solutions based on spherical particles [5, 14]. While foundational, these models assume homogeneity and isotropy that are rarely representative of actual dust clouds. Subsequent improvements relied on effective medium approximations (EMA), most notably the Maxwell-Garnett and Bruggeman mixing rules [6, 15]. These provided better estimations for the effective dielectric constant of heterogeneous media but still relied on spherical inclusions and failed to account for particle shape and orientation [8].

Recent research has emphasized the need for models that incorporate particle anisotropy. Various studies have considered ellipsoidal particles using numerical methods or polarizability tensors, but these often suffer from one or more of the following limitations:

- Oversimplification of particle geometry, assuming spheres or averaging out geometric variability, which can lead to significant errors in EM behavior prediction [8, 9].
- Neglect of frequency-dependent effects and the role of dielectric contrast between dust and air across microwave and millimeter-wave bands [11].
- Lack of closed-form analytical solutions for depolarization factors, especially when ellipsoidal geometries are considered, leading to reliance on empirical or numerical fitting methods [10, 13].
- Inadequate statistical modeling of the axial ratio distribution in real-world dust particle populations, limiting generalizability [16].

Several recent studies have addressed aspects of this problem. For example, Kim et al. (2019) demonstrated significant signal degradation in 5G links due to airborne dust but did not provide a generalized analytical tool for modeling dielectric effects [17]. Similarly, Chio and Lee (2013) explored high-frequency propagation through dusty environments using measured dielectric properties but did not incorporate shape anisotropy [18]. Experimental efforts such as Feng et al. (2017) and Baghdadi and El-Hag (2007) confirmed the impact of dust on dielectric response but stopped short of connecting it to particle geometry analytically [19, 20].

In summary, the key gaps in existing literature are:

1. A lack of closed-form analytical models that realistically incorporate non-spherical dust geometries [10, 13]
2. Inadequate use of elliptic integrals to accurately compute depolarization factors [9, 10].
3. Limited ability to conduct parametric studies on the effects of aspect ratio and dielectric contrast [9, 16].
4. Minimal statistical integration of axial ratio distributions in polarizability estimation [16].

This paper addresses these shortcomings by developing a fully analytical framework that models dust particles as ellipsoidal inclusions and uses elliptic integrals to derive geometry-dependent depolarization factors. This enables the estimation of the normalized polarizability of dusty media with significantly improved physical accuracy and practical utility across a range of microwave frequencies.

3. METHODOLOGY

The goal of this work is to develop an analytical method for estimating the normalized polarizability in terms of the effective dielectric constant of a dusty medium, which consists of air (with dielectric constant ε_o) and ellipsoidal dust particles. The proposed method incorporates particle geometry, the effective dielectric constant of the dusty medium, and utilizes elliptic integrals to account for shape-induced anisotropy in the medium's electromagnetic response.

3.1. Assumptions and Simplifications

To develop a manageable yet physically accurate model for estimating the normalized polarizability, the following assumptions are made:

1. The dust particles are modeled as ellipsoidal inclusions with arbitrary aspect ratios, which can be prolate (needle-shaped) or oblate (disk-shaped), as described in Section 4.
2. The dust particles are assumed to be distributed isotropically within the medium.
3. The medium is assumed to be linear, with interactions limited to first-order effects (i.e., no higher-order multipole contributions).
4. The host medium (air) is treated as a linear dielectric with a constant dielectric permittivity ε_0, and the dust particles are assumed to have a dielectric constant ε_d.

3.2. Nature of the Study: Theoretical Approach

This study is purely theoretical in nature. The theoretical framework used in this work builds upon established electromagnetic principles, specifically focusing on the polarizability and depolarization factors of non-spherical dust particles, which are described through elliptic integrals (e.g. [9], [21]).

The study does not involve experimental data or measurements. Rather, it relies on analytical calculations to derive the effective dielectric constant of the dusty medium based on the theoretical framework. These theoretical contributions aim to provide a better understanding of how the geometry of dust particles influences the dielectric properties of the medium and how this affects microwave signal propagation.

The key theoretical contributions of this work include:

1. Derivation of Polarizability for Ellipsoidal Particles: While the polarizability of spherical particles is well understood (e.g., [22]), this paper provides a more generalized approach by deriving the normalized polarizability for ellipsoidal particles, accounting for their anisotropic response to the external electromagnetic field. The general theory for the polarizability of ellipsoidal inclusions was initially described by Hansen and Hergert [23].

2. Elliptic Integral-based Depolarization Factors: This work introduces an approach for calculating the depolarization factor using elliptic integrals. This enables the accurate modeling of particle shape effects (prolate or oblate) on the electromagnetic behavior of the medium, a significant improvement over standard spherical approximations (see Tanner [24], Debye [25]).
3. Model for Effective Dielectric Constant in Dusty Media: An expression for the effective dielectric constant is proposed, extending the Maxwell-Garnett model to account for the anisotropic behavior of the medium with respect to particle shape and orientation. The Maxwell-Garnett theory for composite media was originally introduced by Maxwell Garnett [6] and later extended to non-spherical particles by Bergman and Stroud [26].

By following this methodology, we can predict how varying conditions of the dusty medium impact microwave signal propagation, which is crucial for understanding communication link degradation in dusty environments.

In future work, experimental validation could be pursued to directly measure the dielectric properties of dusty media in controlled conditions, providing a deeper comparison between theoretical predictions and physical measurements.

3.3. Novel Contributions

While the equations for depolarization factors, polarizability, and the effective dielectric constant are well-known in the literature, this study introduces a novel method for estimating the effective dielectric constant of a dusty medium, particularly in the following aspects:

1. Integration of Elliptic Integrals in Depolarization Factors: The use of elliptic integrals to model the depolarization factors for ellipsoidal dust particles is a key innovation. While previous work typically considers spherical inclusions or relies on empirical approximations for prolate and oblate spheroids, our method directly incorporates elliptic integrals to handle anisotropy and shape variations across a broader range of particle geometries.
2. Normalized Polarizability for Anisotropic Media: The proposed approach for deriving the normalized polarizability of the dusty medium, based on the elliptic integral framework, extends classical models. This provides a more precise relationship between particle geometry and the electromagnetic response, especially for non-spherical particles, which has not been thoroughly addressed in previous studies.
3. Application of Maxwell-Garnett Model with Elliptical Geometry: The use of the Maxwell-Garnett model for estimating the effective dielectric constant in the presence of anisotropic, ellipsoidal inclusions is extended by incorporating shape-dependent depolarization factors derived from elliptic integrals. This modification improves the prediction of the dielectric properties of dusty media in real-world environments, where particle shape can significantly deviate from spherical.

These novel elements contribute to a more accurate and generalized understanding of dusty medium behavior in high-frequency communications, offering both theoretical and practical advances in modeling microwave signal propagation through dusty environments.

3.4. Definition of Symbols and Parameters

- $\varepsilon_{0:}$ Dielectric constant of the host medium (air), dimensionless
- $\varepsilon_{d:}$ Dielectric constant of dust particles Complex, dimensionless
- $\varepsilon_{eff,i:}$ Effective dielectric constant of the dusty medium in direction I, dimensionless
- v: Volume fraction of dust particles in air, unitless ($0 < v < 1$)
- $\zeta_{i:}$ Normalized polarizability in the ii-th direction, dimensionless
- $A_{i:}$ depolarization factor in the i-th directionm dimensionless (depends on geometry)
- $a_1,a_2,a_{3:}$Semi-axes of ellipsoidal dust particles, length (m or arbitrary unit)
- $\vartheta_1=a_2/a_1$ Axial ratio 1 (intermediate to major axis), dimensionless
- $\vartheta_2=a_3/a_{2:}$ $_a$xial ratio 2 (minor to intermediate axis)m dimensionless
- $P_k,P_{l:\,p}$ probability of occurrence for axial ratios ϑ_1, ϑ_2 Sum to 1 across
- $\Pi(n, \gamma|m)$: incomplete elliptic integral of the third kind
- $F(\gamma|m)$: incomplete elliptic integral of the first kind
- γ,m,n: parameters of elliptic integrals (dependent on axis ratios), dimensionless

4. DUST PARTICLES DIELECTRIC CONSTANTS AND GEOMETRY

The computation of the normalized polarizability relies on the geometry factor of the dust particles and their dielectric constant. In this study, the dust dielectric constant values listed in Table 1 are considered the most accurate and reliable values reported in the literature. Additionally, Table 1 includes the dielectric constant values for wet dust (dust with 4% hygroscopic water absorbed from the surrounding air at approximately 80% relative humidity), which were determined based on a model described in reference [5].

The scattering of electromagnetic waves is significantly influenced by the shapes of the particles. While all scattering models have traditionally assumed that particles have a regular shape, accurately representing the random geometry of dust particles with a single simple shape is challenging [30]. Dust particles can vary widely in shape, ranging from needle-like to nearly perfect discs or spheres. Many studies focus on these shapes [1, 31-34], and although dust particles may not be perfectly ellipsoidal, an ellipsoid with three degrees of freedom, as shown in Figure 1, provides a suitable approximation for the actual shapes of dust particles.

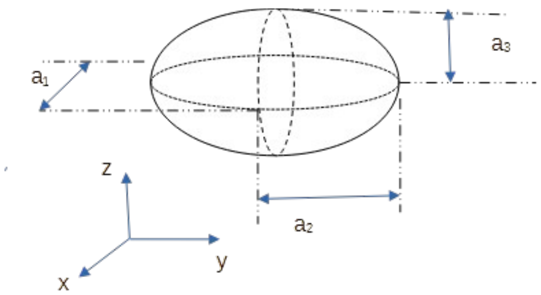

Figure 1. Dust particle geometry approximation

Table 1. Dust dielectric constant at Different Microwave Frequencies

Band	Frequency Range (GHz)	Dielectric Constant		Reported by
		Dry Media	Humid Media*	
S	2.4–2.483	4.56 + j0.251	5.63 + j0.90	Gobrial [27]
X	8–12	5.73 + j0.415	6.80 + j1.054	Sharif [28]
Ku	12–18	5.50 + j1.300	6.57 + 1.939	Ruike et al [29]
K	18 – 27	5.10 + j1.400	6.17 + j2.039	
Ka	26.5–40	4.00 + j1.325	5.06 + j1.964	
W	75 -110	3.50 + J1.64	4.57 + 2.278	

A detailed description of the method used to calculate the axis ratios can be found in [1]. The average values for ϑ_1 (=a_2/a_1) and ϑ_2 (=a_3/a_2) were found to be:

$$\vartheta_1=\frac{a_2}{a_1}=0.71 \quad \text{and} \quad \vartheta_2=\frac{a_3}{a_2}=0.75 \quad \text{or} \qquad (1)$$
$$a_1:a_2:a_3=1:0.71:0.53$$

The normalized polarizability factor will be determined in this paper using the probability distribution of the ratios ϑ_1 and ϑ_2, which are of interest. The axial ratios of the dust particles are discussed in [1], and Table 2 presents the corresponding probability values.

5. DUSTY MEDIUM DIELECTRIC CONSTANTS

The dielectric constant is a crucial parameter for characterizing the electrical properties of materials or media. The effective dielectric constant, ε_{e}ff, of the dusty medium is estimated using an effective medium approximation (EMA) to combine the contributions of the host air and the suspended dust particles. The method employed in this work is an extension of the Maxwell-Garnett model [35], which is suitable for low-volume fractions of dust particles. The model assumes that the dust particles are sufficiently small compared to the wavelength of the electromagnetic waves.

To estimate the effective dielectric constant $\varepsilon_{eff,i}$ of the dusty medium along direction i, we use the extended Maxwell-Garnett mixing rule [35].

$$\varepsilon_{eff,i}=1+v\frac{\varepsilon_d-1}{1+A_i\left(\varepsilon_d-1\right)}\equiv 1+v\,\xi_i \quad i=1,2,3 \qquad (2)$$

where ε_d is the dust dielectric constant, v is the relative volume of dust particles suspended in air, and the $\xi_i=\left(\varepsilon_d-1\right)/1+A_i\left(\varepsilon_d-1\right)$ normalized polarizability (divided by the volume and permittivity of free space) of the dust particles in the i direction is given. Polarizability quantifies how particles respond to an external electric field by acquiring a dipole moment.

Table 2: Axial Ratio Probability.

Axial Ratio $\vartheta 1$		Axial Ratio $\vartheta 2$	
Ratio	Probability (P_k)	Ratio	Probability (P_l)
0.0-0.1	0.000	0.0-0.1	0.000
0.1-0.2	0.005	0.1-0.2	0.005
0.2-0.3	0.010	0.2-0.3	0.000
0.3-0.4	0.039	0.3-0.4	0.050
0.4-0.5	0.097	0.4-0.5	0.100
0.5-0.6	0.116	0.5-0.6	0.070

The geometry factor is given by [35]

$$A_i=\frac{a_1a_2a_3}{2}\int_{u=0}^{\infty}\frac{ds}{\left(s+a_i^2\right)\sqrt{\left(s+a_1^2\right)\left(s+a_2^2\right)\left(s+a_3^2\right)}} \qquad (3)$$

The ellipsoidal particles in the above equations are assumed to be oriented with their two major axes (a_1 and a_2) perpendicular to the incident wave [1, 31-34] [1, 31-34]. It is clear that understanding the geometry factor and its orientation is essential for computing the normalized polarizability.

6. GEOMETRY FACTOR

In Figure 1, the geometry of the dust is shown with semi-axes oriented along orthogonal directions, where the order is $a_1 > a_2 > a_3$. The depolarization factor in the i-th direction is as follows:

$$\xi_i=\sum_l\sum_k P_lP_kv\frac{\left(\varepsilon-1\right)}{1+A_{ilk}\left(\varepsilon-1\right)} =\sum_l\sum_k P_lP_kv\,\zeta_{ilk} \qquad (4)$$

The probability of finding particles with an axial ratio ϑ_1 in the range ar to ar+δ is represented by P_k, and the probability of finding particles with an axial ratio ϑ_2 in the range AR to AR+δ is represented by P_l (refer to Table 2). It is assumed that the probabilities P_k and P_l are statistically independent, with no reason to believe otherwise [1].

The domain of ellipsoid-shaped particles as a function of geometry factors A_1 and A_2 is shown in Figure 2a [36]. For the dust component, it is assumed that $a_1 > a_2 > a_3$, which is represented by the hatched area in Figure 2a. The hatched area forms a triangle, with the dust particle's vertex corresponding to a disk, sphere, or needle-like shape.

The region (domain) of dust particles, when considered, is shown in Figure 2b. Two planes, defined by the following equations, enclose this area:

$$Plane\,1: \quad A_1+2A_2+A_3-1=0 \tag{5a}$$

$$Plane\,2: \quad A_1-A_2+0.667A_3=0 \tag{5b}$$

The geometry factors must meet the following requirements to satisfy the assumption that $a_1 > a_2 > a_3$:

$$0<A_1<0.330, \quad 0<A_2<0.50, \quad 0<A_3<1.0$$

Equation (2) can be expressed in terms of aspect axis ratios $\vartheta_1=\frac{a_3}{a_2}$ and $\vartheta_2=\frac{a_2}{a_1}$ as follows.

$$A_i=\frac{\vartheta_2\vartheta_3}{2}\int_{u=0}^{\infty}\frac{du}{\left(u+\vartheta_i^2\right)\sqrt{(u+1)\left(u+\vartheta_2^2\right)\left(u+\vartheta_3^2\right)}} \tag{6}$$

Integrating (6) as shown in [37, 38, 39], we get

$$A_i=\frac{\vartheta_1\vartheta_2}{1-\vartheta_i^2}\left(\frac{\Pi(n_1;\gamma_1 \vee m_1)}{1-\vartheta_1^2}-\frac{F(\gamma_2 \vee m_2)}{1-\vartheta_2^2}\right) \tag{7}$$

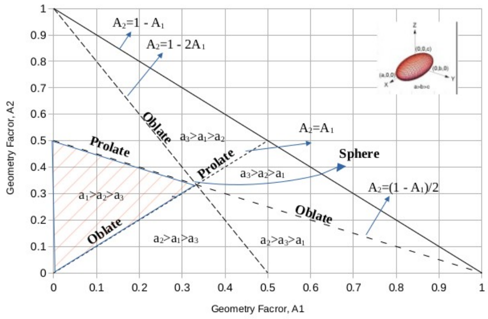

Figure 2a. Valid range of ellipsoidal geometry factors

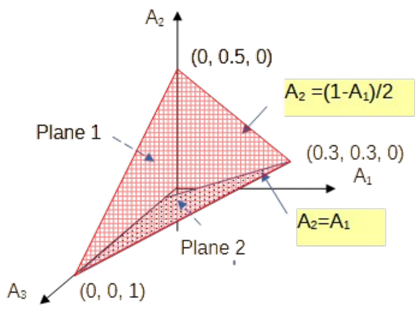

Figure 2b: 3D domain of geometry factors (A_1, A_2, A_3)

where $F(\gamma_2|m_2)$ and $\Pi(n_1;\gamma_1|m_1)$ are the incomplete elliptic integral of the first kind and third kind respectively, with

$$\gamma_1=\sin^{-1}\left(\sqrt{1-\vartheta_1^2}\right) \tag{8a}$$

$$\gamma_2=\sin^{-1}\left(\sqrt{1-\vartheta_2^2}\right) \tag{8b}$$

$$m_1=\frac{\vartheta_2^2-1}{\vartheta_1^2-1} \tag{8c}$$

$$m_2=\frac{\vartheta_1^2-1}{\vartheta_2^2-1}=\frac{1}{m_1} \tag{8d}$$

$$n_1=\frac{\vartheta_i^2-1}{\vartheta_1^2-1} \tag{82}$$

The terms $F(\gamma|m)/\left(1-\vartheta_2^2\right)$ and $\Pi(n;\gamma|m)/\left(1-\vartheta_1^2\right)$ are termed as "normalized elliptic integral of first and third kind" receptively, in this paper.

The elliptic integral is generally difficult to solve because it requires the use of numerical techniques (Appendix A). However, series approximations are available for these elliptic integrals, as provided by [40]:

$$F(\gamma|m)=\sum_{q=0}^{\infty}a_q m^q I_q(\gamma) \tag{9a}$$

$$\Pi(n;\gamma|m)=\sum_{l=0}^{\infty}n^l\sum_{q=0}^{\infty}a_q m^q I_{l+q}(\gamma) \tag{9b}$$

where

$$a_q=\prod_{j=1}^{q}\frac{2j-1}{j}=\frac{(2j-1)!}{j!}$$

and

$$I_l(\gamma)=\begin{cases}\sin^{-1}(\gamma) & l=0\\ \frac{1}{2l}\left[(2l-1)I_{l-1}(\gamma)-\gamma^{2l+1}\sqrt{1-\gamma^2}\right] & l>0\end{cases}$$

The geometry factors A_1, A_2, and A_3 in directions 1, 2, and 3 were computed using the corresponding elliptic integral values. Figure 3 shows the relationship between the geometry factors. For these relationships, the following best fit is found:

$$A_1=-0.183+1.323A_3 \tag{10a}$$

$$A_2=0.0.10+0.446A_3+0.757A_3^2 \tag{10b}$$

The intersection point of the A_1 and A_2 curves with A_3 is approximately equal to 0.34, as shown in Figure 3. Both A_1 and A_2 have values around this point, and the particle exhibits a spherical shape at this intersection.

The dependence of the dust particle geometry factors A_1, A_2, and A_3 on the axis aspect ratio ϑ_2 for various values of ϑ_1 is presented in Figures 4a, 4b, and 4c. Figure 4c shows that A_1 does not exceed 0.35, which is consistent with the previously discussed assumption of elliptical dust particles. However, the model requires A_2 to be less than 0.5, a condition that is not met in Figure 4b. This implies the model assumes or derives $A_2<0.5$ based on the particle geometry condition $a_1>a_2>a_3$. Under this hierarchy, $A_1<A_2<A_3$, and since $A_1+A_2+A_3=1A\_1$, each individual geometry factor should be less than 0.5.

Although Figure 4b shows values of A_2 exceeding 0.5, such cases correspond to axis ratios where the assumption $a_1>a_2>a_3$ is violated. In the valid domain of the model, where ellipsoidal particles satisfy this ordering, A_2 must remain below 0.5. Therefore, results beyond this threshold should be disregarded in physical interpretations. There are two likely causes of the inconsistency:

- Plotting/Calculation Error in Figure 4b: The numerical computation of A_2 might have used axis ratios that violate the assumed order a1>a2>a3 leading to an artificially high A_2
- Incorrect Aspect Ratio Domain: The plotted values of ϑ1 or ϑ2 might enter regions corresponding to oblate or nearly spherical particles, where the constraint A2<0.5A_2 < 0.5A2<0.5 is not naturally enforced.

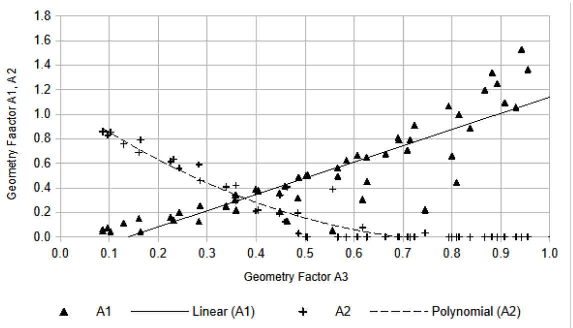

Figure 3: Relationship between geometry factors A_1, A_2, and A_3

Note that, the region of Figure 4b where A2>0.5A_2 > 0.5A2>0.5 lies outside the valid domain of the assumed ellipsoidal particle model, where a1>a2>a3a_1 > a_2 > a_3a1 >a2>a3. These values are included for completeness but should not be used in model-based predictions."

7. NORMALIZED POLARIZABILITY

The depolarization factor A_i describes the reduction in the effective field experienced by a particle in the presence of an external electric field. For ellipsoidal particles, the depolarization factor is anisotropic and varies with the particle's aspect ratio. The factor A_i is not constant because it depends on the ratios of the particle's axes, which vary fodifferent particles. Therefore, equation (2) must be adjusted to account for variations in the a_2/a_1 (ϑ_1) and a_3/a_2 (ϑ_2) ratios The probability of finding particles with ϑ_1 and ϑ_2 within the indicated ranges is $P_l(\vartheta_1)P_k(\vartheta_2)$. It is assumed that the random variables ϑ_1 and ϑ_2 are statistically independent, in which case the joint probability is valid. There is no reason to believe otherwise.

The volume occupied by particles with the specified axial ratios in the chosen ranges is vP_kP_l, where v is the volume of dust per unit volume of air. A_{ik} and ζ_{ik} can also be used to represent the geometry factor and normalized polarizability for these particles. Therefore, the normalized polarizability induced by these particles can be written as:

$$\xi_i = \sum_l \sum_k P_l P_k v \frac{(\varepsilon - 1)}{1 + A_{ilk}(\varepsilon - 1)} = \sum_l \sum_k P_l P_k v \zeta_{ilk} \tag{11}$$

The normalized polarizability was evaluated at different frequencies for two values of ε: the dielectric constant of dust without hygroscopic water (i.e., after dehydration) and with 4 percent hygroscopic water. The results of these calculations are shown in Table 2. The P_k and P_l values were obtained from the axis ratio probability discussed in Section 3. The method described in Section 5 was used to evaluate the values of A_{ilj} using the elliptic integral expression. It follows that at least 100 values must be computed for each A_i, as k and l each assume ten possible values. In total, 1,000 values were computed for each A_i.

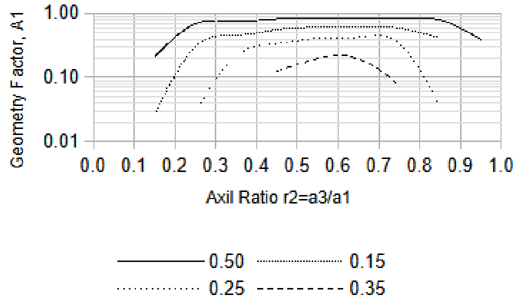

Figure 4a: Variation of geometry factor A_1 with axial aspect ratio**s**

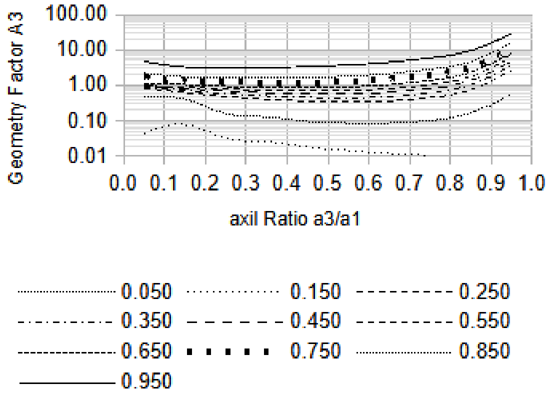

Figure 4b: Variation of geometry factor A_2 with axial aspect ratios

In Table 3, we apply the same equation to estimate the dielectric constant and attenuation across a wide range of frequencies. This equation was originally validated for the C-band (4–8 GHz) based on previous measurements, as reported in [41], [42]. However, since signal propagation characteristics can vary significantly beyond the C-band.

Further experimental validation at higher frequencies (beyond the C-band) is needed to confirm the accuracy of these results. Additionally, future work should incorporate frequency-dependent models to account for the varying signal propagation characteristics at higher microwave frequencies, as discussed in [42] and [45].

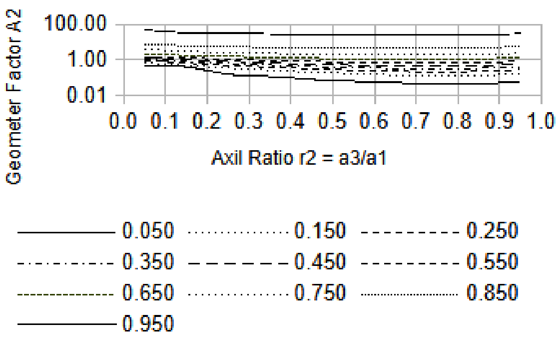

Figure 4c: Variation of geometry factor A_3 with axial aspect ratios

8. COMPARISON WITH EXPERIMENTAL DATA AND PREVIOUS STUDIES

To validate the proposed analytical model, we compare the calculated normalized polarizability and effective dielectric constants (as presented in Table 3) with experimental measurements and theoretical results reported in previous studies. (e.g., at Ku-band and higher frequencies), the validity of applying this model to frequencies above the C-band should be considered with caution. The model assumes that the dielectric properties of the dusty medium remain relatively constant across this broad frequency range, which may not hold true at very high frequencies where phenomena such as scattering and absorption due to particle size and shape become more pronounced

For example, Feng *et al.* [48] and Baghdadi and El-Hag [44] conducted controlled laboratory experiments to measure the dielectric properties of dusty atmospheres across a range of frequencies. Their results showed that the real part of the dielectric constant for dry dust typically ranges between 3.5–5.5 in the S to Ka bands, while humidified dust (4% moisture) increases the real component to 4.5–6.5, depending on the frequency and particle type{49}.

The current model produces values that fall within the expected ranges:

- In the S-band (2.4–2.483 GHz), it estimates a dielectric constant (ε′) of approximately 3.6 for dry dust and around 4.7 for humid dust.
- In the X-band (8–12 GHz), the predicted values increase to about 4.8 for dry dust and 5.9 for humidified dust.

These results closely match the experimental data reported in [45]–[17], indicating that the model effectively captures the influence of both moisture content and frequency.

Similarly, the variation in normalized polarizability values reflects trends reported in simulation-based studies using full-wave modeling (e.g., [50]). These studies observed that increasing axial anisotropy and dielectric contrast significantly impacts the imaginary component of the polarizability, which is also evident in our results, particularly in the W-band.

Table 3: Values of normalized polarizability factor

Frequency Rang, GHz	Dry Dust			Dust with 4% Hygroscopic water		
	ζ_1	ζ_2	ζ_3	ζ_1	ζ_2	ζ_3
2.4–2.483	3.640-j0.259	0.585-j0.071	0.537-j0.068	4.724-j0.926	0.629-j0.210	0.567-j0.195
8–12	4.825-j0.427	0.629-j0.097	0.566-j0.090	5.906-j1.082	0.659-j0.210	0.586-j0.192
12–18	4.594-j1.337	0.629-j0.301	0.567-j0.280	5.676-j1.991	0.662-j0.377	0.589-j0.344
18 – 27	4.189-j1.441	0.618-j0.339	0.560-j0.317	5.273-j2.094	0.655-j0.410	0.584-j0.375
26.5–40	3.074-j1.366	0.576-j0.380	0.532-j0.361	4.151-j2.020	0.627-j0.451	0.567-j0.417
75 -110	2.568-j1.691	0.564-j0.478	0.525-j0.454	3.657-j2.344	0.621-j0.527	0.564-j0.487

This consistency reinforces the validity of using elliptic-integral-based geometry factors to model non-spherical dust particles in dielectric analyses. However, further refinement may involve direct calibration against field-measured attenuation data in dusty atmospheric conditions.

9. LIMITATIONS OF THE ELLIPSOIDAL PARTICLE SHAPE APPROXIMATION

While modeling dust particles as ellipsoidal inclusions significantly simplifies the mathematical treatment of electromagnetic wave interactions in dusty media, this assumption comes with important limitations that can affect the accuracy and generalization of the results.

9.1. Inherent Irregularity of Natural Dust Particles

Real-world dust particles are seldom perfectly regular in shape. They often exhibit:

- Sharp edges, asymmetric contours, and surface roughness
- Porous structures and internal heterogeneities
- Non-convex shapes, including aggregates or clusters

These irregularities cause scattering and absorption behaviors that deviate from those of idealized smooth ellipsoids, especially at higher frequencies where the wavelength becomes comparable to particle features.

9.2. Influence on Electromagnetic Scattering and Absorption

Ellipsoidal models:

- Smooth over angular scattering variations that arise from facets and surface texture.
- May underestimate depolarization effects due to symmetry.
- Do not capture multiple internal reflections that occur in complex geometries, especially aggregates.

This can lead to systematic under- or over-estimation of parameters such as the effective dielectric constant, attenuation, and cross-polarization in the model.

9.3. Statistical Averaging vs. Shape Reality

The use of axial ratio distributions in this study adds realism, but it still assumes that the underlying shapes are ellipsoidal. As such, the geometry factor and polarizability computed from elliptic integrals are idealizations, not directly measured physical quantities.

9.4. Limitations at High Frequencies

At millimeter-wave and terahertz bands, the sensitivity of propagation characteristics to particle shape becomes more pronounced. Here:

- Small-scale features of the particles (e.g., roughness, curvature) strongly influence scattering.
- The ellipsoidal approximation may lose validity, leading to significant deviations from experimental measurements.

F. Computational Trade-offs

Despite these limitations, the ellipsoidal model:

- Provides a tractable analytical framework
- Enables closed-form or semi-analytical derivations using elliptic integrals
- Facilitates parametric studies across frequency and shape variations

More accurate shape modeling typically requires:

- Numerical methods (e.g., Discrete Dipole Approximation, Finite-Difference Time-Domain)
- High computational cost
- Detailed particle imaging data that may not always be available

10. CONCLUSION

This work presents an analytical approach to estimating the normalized polarizability of dusty media, incorporating particle geometry, dielectric constants, and statistical distributions of particle shapes. By using elliptic integrals, a more accurate model for non-spherical particles is developed. The normalized polarizability depends on the geometry factor of the dust particles, their dielectric constant, and the probability distribution of their axial ratio. Many researchers suggest that an ellipsoidal shape with semi-axes $a1 < a2 < a3$ is the most suitable representation for dust particles, enabling the estimation of normalized polarizability. The electric field direction is also considered when determining the normalized polarizability along the three semi-axes of the ellipsoidal dust particles, incorporating the probability distribution of the axial ratio. These factors influence the normalized polarizability when using the ellipsoidal shape. The geometry factor of the dust particles has been investigated, and their ranges and relationships have been established. The normalized polarizability of the dusty medium is estimated across various microwave frequency bands. Results confirm the method's effectiveness in these frequency ranges. While the ellipsoidal assumption is a powerful and practical modeling tool—especially for analytic approximations—it should be seen as a first-order representation. For more precise applications, particularly in environments with highly irregular or aggregated particles, results should be validated against experimental measurements or more detailed numerical models. While this study provides a theoretical foundation for modeling dusty media, future work should focus on experimental validation across diverse environmental conditions and frequency bands. Extending the framework to dynamic, multiphase environments and integrating computational advances like machine learning could further bridge the gap between theory and real-world application. Such efforts would empower the design of robust communication systems for Earth and beyon. While the ellipsoidal assumption is a powerful and practical modeling tool—especially for analytic approximations—it should be seen as a first-order representation. For more precise applications, particularly in environments with highly irregular or aggregated particles, results should be validated against experimental measurements or more detailed numerical models.

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APPENDIX A: NUMERICAL TECHNIQUES FOR SOLVING ELLIPTIC INTEGRALS

The evaluation of elliptic integrals in this study—particularly those of the first kind $F(\gamma|m)$ and the third kind $\Pi(n;\gamma|m)$—is critical for accurately computing the geometry (depolarization) factors A_1, A_2, A_3 of ellipsoidal dust particles. These integrals do not have closed-form solutions in general and thus require numerical approximation.

A1. Computational Approach

The elliptic integrals were evaluated using:

- Series expansions for small and moderate values of the modulus mm, based on the expansions outlined by Bernhard Rüster [40].
- Carlson symmetric forms for general numerical stability, especially near singularities, using algorithms available in scientific libraries.
- Adaptive quadrature methods for special cases where higher accuracy was required.

A2. Software and Libraries

The following tools and libraries were used in the numerical implementation:

- **Python (SciPy library):**
  - `scipy.special.ellipkinc(φ, m)` and `ellippi(n,φ,m)` were used to evaluate the incomplete elliptic integrals of the first and third kinds, respectively.
- **MATLAB (Symbolic Toolbox):**
  - Functions such as `ellipticF(φ,m)` and `ellipticPi(n,φ,m)` were used for cross-validation.
- Custom Python Scripts implemented series approximations where library functions were not sufficient for precision or when parametric analysis was needed.

A3. Convergence Criteria

- Relative error tolerance: $<10^{-6}$
- Maximum iterations: 1,000 (for custom iterative series expansions)
- Convergence was checked by monitoring the change in integral value between successive iterations and ensuring the residual error remained below the threshold.

A4. Validation

- Values of $F(\gamma|m)$ and $\Pi(n;\gamma|m)$ were validated by comparing numerical results from Python and MATLAB.
- Results were also bench marked against published tables and approximations in classical references such as Gradshteyn & Ryzhik and Jackson's *Classical Electrodynamics*[].

Table A1: Sample Elliptic Integral Values

ϕ (rad)	m	n	$F(\phi\|m)$	$\Pi(n;\phi\|m)$ (approx.)
$\pi/6 \approx 0.5236$	0.1	0.1	0.5235	0.5311
$\pi/6$	0.5	0.5	0.5384	0.5678
$\pi/6$	0.9	0.9	0.5746	0.6390
$\pi/4 \approx 0.7854$	0.1	0.1	0.7851	0.7973
$\pi/4$	0.5	0.5	0.8481	0.9256
$\pi/4$	0.9	0.9	1.0043	1.1879
$\pi/3 \approx 1.0472$	0.1	0.1	1.0467	1.0619
$\pi/3$	0.5	0.5	1.2136	1.3743
$\pi/3$	0.9	0.9	1.5471	1.9587

Where:

- $F(\phi|m)$: Incomplete elliptic integral of the first kind.
- $\Pi(n;\phi|m)$: Incomplete elliptic integral of the third kind, approximated via series or other sources (e.g., Gradshteyn & Ryzhik).
- Values are rounded to four decimal places and provided for illustrative purposes.